Model Theory - Lecture 4 - Quantifier elimination

Written Exam 14/10

Two examples

a formula in the language of fields (or rings)

- $\exists x \, (x^2+bx+c=0) \equiv \varphi$

  This formula is true (i.e, the polinomial has a solution)

  in $\mathbb{R}$ iff $b^2-4c \geq 0$.

  also a formula but in the language of ordered fields. Also, it is quantifier-free.

  $\mathcal{T}(\mathbb{R})$ eliminates the quantifier for φ
  (in the language of ordered fields)

- $\exists$ non-zero solution of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \equiv \psi$

  Δ(matrix)

  But then, again, we can just ask $ad-bc \neq 0$

<u>Definition</u> A theory $\mathcal{T}$ eliminates quantifiers iff, for every formula φ, $\mathcal{T}$ proves

$$\mathcal{T} \models \forall x \, (\varphi(x) \longleftrightarrow \psi(x)),$$

where ψ has no quantifiers

We give a charachterization of this definition, but we need ano-
ther concept

<u>Definition</u> A formula is "primitive" if it is of the form

$\exists x\, \varphi(x)$ where φ is a conjunction of atomic formulas or its negation

<u>Theorem</u> A theory (in FOL!) admits quant elim QE iff it admits

it for all primitive formulas

<u>Proof</u> By induction on the complexity of the formula (for the non-trivial case) and using the disjunctive normal form to reduce everything to the primitive case

Notice $\exists x\,(\varphi \vee \psi) \longleftrightarrow \exists x \varphi \vee \exists x \psi$

∎

<u>Remark</u> If in your logic $\exists$ and $\vee$ don't distribute, then the proof will <u>not</u> work. In general, the theorem is not neces-
sary true in other logics

We can give the following necessary condition

<u>Theorem</u> If a theory eliminates quantifiers, then every embedding is elementary (model complete), i.e. if $\mathcal{M} \hookrightarrow \mathcal{N}$ is an embedding of models in said theory, then it is elementary ($\mathcal{M} \models \varphi$ iff $\mathcal{N} \models \varphi$)

the proof follows from the previous theorem.

<u>Remark</u> this is far from being a sufficient condition

<u>Remark:</u> the theory of fields does <u>not</u> eliminate quantifiers, since $\mathbb{Q}$ can be embedded in $\mathbb{R}$ but never 'elementarily' as some of the roots do lay in $\mathbb{R} \setminus \mathbb{Q}$

We will now start to develop the technique to get a sufficient condition as well (we won't see it today, but next lecture).

BACK and FORTH (Saturated Models) general idea can I scale up finite configurations to build models?

<u>Definition</u> A "partial isomorphism between models $\mathcal{M}$ and $\mathcal{N}$" is a function

$$f: \underset{\mathcal{M}}{\overset{\cap}{A}} \longrightarrow \underset{\mathcal{N}}{\overset{\cap}{B}},$$ where A, B are subsets

if, for every atomic formula φ,

$\mathcal{M} \models \varphi(a_i)$ iff $\mathcal{N} \models \varphi(f(a_i))$, with $a_i \in A$

<u>Remark</u>: $\underset{\mathcal{M}}{\overset{\cap}{\phi}} \xrightarrow{f} \underset{\mathcal{N}}{\overset{\cap}{\phi}}$ is not always a partial isomorphism

For example $\underset{\mathbb{F}_2}{\overset{\cap}{\phi}} \xrightarrow{f} \underset{\mathbb{F}_3}{\overset{\cap}{\phi}}$ with $\varphi \equiv 1+1=0$ Notice φ has no free variables

<u>Proposition</u> if $\underset{\mathcal{M}}{\overset{\cap}{A}} \xrightarrow{f} \underset{\mathcal{N}}{\overset{\cap}{B}}$ is a bijective function it is a partial isomorphism iff it extends to an isomorphism of generated structures. ← (structure, not model!)

<u>Proof</u> →) There's only one way to build the extension, and it is easy to check it is an isomorphism

←) Just need to check the atomic formulas

$\blacksquare$

Definition $\underset{\mathcal{M}}{\overset{\text{\scriptsize a collection of functions}}{A}} \xrightarrow{I} \underset{\mathcal{N}}{B}$ With the data here we say that the collection I "has the back and forth (property)" iff

- all $f \in I$ are partial isomorphisms between $\mathcal{M}$ and $\mathcal{N}$
- for all $f \in I$ and all $m \in \mathcal{M}$, there exists $g \in I$ such that $f \subseteq g$ and $m \in \operatorname{dom}(g)$
- for all $f \in I$ and all $n \in \mathcal{N}$, there exists $g \in I$ such that $f \subseteq g$ and $n \in \operatorname{rng}(g)$

When such an I exists, we write $\mathcal{M} \equiv_I \mathcal{N}$ The name is not very established, we will use "$\mathcal{M}$ is partially isomorphic to $\mathcal{N}$"

Example In the theory of dense linear orders without endpoints. Consider I the collection of all finite partial isomorphisms between $\mathbb{Q}$ and $\mathbb{R}$ Then, I has the B&F, so $\mathbb{Q} \equiv_I \mathbb{R}$ in this theory But if we choose $\mathbb{Z}$ and $\mathbb{R}$ we do not have B&F with the "same" I

Theorem (Scott) $\mathcal{M} \equiv_p \mathcal{N}$ and $\mathcal{M}, \mathcal{N}$ are countable, then they meaning partially iso are isomorphic

Proof We just build it! We start from $\underset{\mathcal{M}}{\underset{\cap}{\phi}} \dashrightarrow \underset{\mathcal{N}}{\underset{\cap}{\phi}}$ and proceed by induction

- on even steps we extend $\mathcal{M}$;
- on odd steps we extend $\mathcal{N}$.

At the end, we just take the union. Easy to check it's iso ∎

Corollary (Cantor) the theory of dense linear orders has exactly one countably infinite model

Proof The I from the example always has the B&F on that ∎

(More on this? Classification theory / Morley Categoricity theorem)